Trisymmetric multiplication formulae in finite fields

Hugues Randriambololona, Édouard Rousseau

July 4, 2020

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  - $O(n \log n)$ algorithm [Harvey, Van Der Hoeven '19]

Bilinear complexity: intuition

- $\mathcal{A}$ an algebra over $\mathbb{K}$
- **bilinear complexity:** number of subproduct in $\mathbb{K}$ needed to compute a product in $\mathcal{A}$

Karatsuba:

$$(a_0 + a_1 X)(b_0 + b_1 X) =$$

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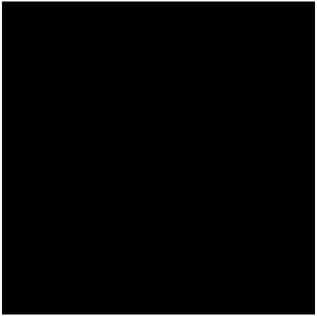
- ☹ **Hard** to compute the bilinear complexity of a product: unkwown even for the 3×3 matrix product

Complexity of Karatsuba's algorithm



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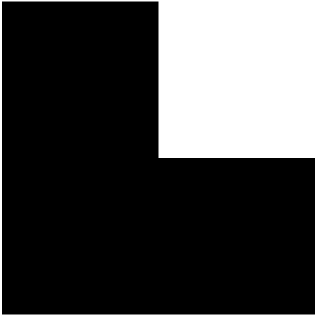

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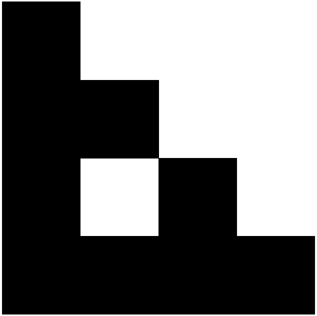

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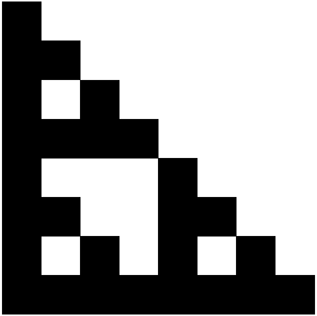

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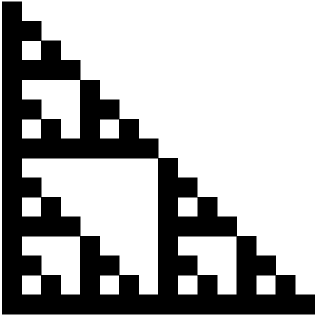

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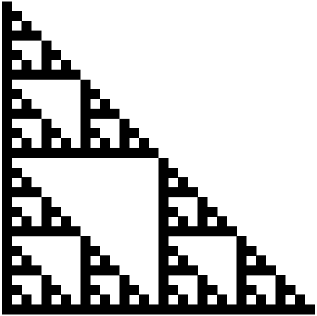

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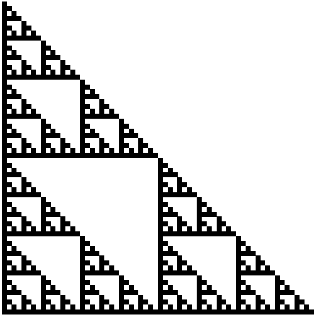

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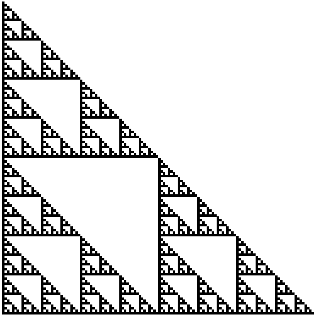

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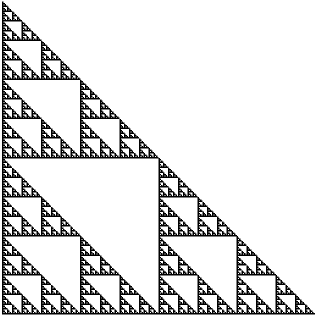

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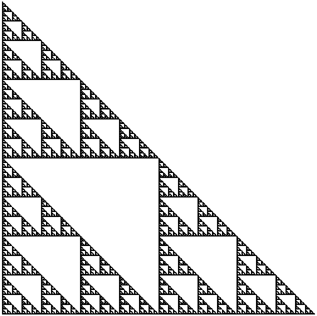

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Bilinear complexity: definition

Definition

The **bilinear complexity** of the product in $\mathcal{A}$ is the minimal integer $r \in \mathbb{N}$ such that you can write, for all $x, y \in \mathcal{A}$

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Notations and questions

- $\mu_q(m)$ = bilinear complexity of the product in $\mathcal{A} = \mathbb{F}_{q^m}$

Two independent questions:

- What is the asymptotic comportment of $\mu_q(m)$?

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    - [Randriambololona '12]
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  - Clever exhaustive search [BDEZ '12] [Covanov '18]

EVALUATION-INTERPOLATION SCHEMES

Karatsuba again:

- $P(X) = a_0 + a_1 X, Q(X) = b_0 + b_1 X$

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- When studying $\mathcal{A} = \mathbb{F}_{q^m}$ for $m \to \infty$, one needs **many points** of evaluation

$\rightsquigarrow$ use a **curve** on $\mathbb{F}_q$ with **many points** for evaluations

Symmetric decompositions

- $\mathcal{A}$ **commutative** algebra

Classic decompositions	**Symmetric** decompositions
$xy = \sum_{j=1}^{r} \varphi_j(x)\psi_j(y) \cdot \alpha_j$	$yx = xy = \sum_{j=1}^{r} \varphi_j(x)\varphi_j(y) \cdot \alpha_j$

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- **Small values: smaller** search space $\rightsquigarrow$ **faster** algorithms

Even more symmetric decompositions

- $\mathcal{A} = \mathbb{F}_{q^m}$
- every linear form φ can be written $x \mapsto \mathrm{Tr}(\alpha x)$ for some $\alpha \in \mathbb{F}_{q^m}$, with Tr the trace of $\mathbb{F}_{q^m}/\mathbb{F}_q$
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- we note $\mu_q^{\mathrm{tri}}(m)$ the minimal r in such formulae

Example of trisymmetric decomposition

- $\mathcal{A} = \mathbb{F}_{3^2} \cong \mathbb{F}_3[z]/(z^2 - z - 1) \cong \mathbb{F}_3(\zeta)$
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About trisymmetric decompositions

Link with other decompositions:

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Tri-symmetric decompositions always exist, except for $q = 2, m \geq 3$.

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- **Assymptotics:** linearity in m can be obtained for **symmetric** decomposition in $\mathbb{F}_{q^m}$ in **higher dimensions**
  - Corollary: $\mu_q^{\text{tri}}(m)$ is also **linear** in m

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Proposition (Randriambololona, '14)

Tri-symmetric decompositions always exist, except for $q = 2, m \geq 3$.

- **Assymptotics:** linearity in m can be obtained for **symmetric** decomposition in $\mathbb{F}_{q^m}$ in **higher dimensions**
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  - We provide an *ad hoc* exhaustive search algorithm

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Thank you!