Trisymmetric multiplication formulas in finite fields

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Séminaire CRYPTO

FINITE FIELDS IN CRYPTOGRAPHY

Finite fields are (almost) everywhere in **public key** cryptography:

- discrete logarithm
- elliptic curves
- isogenies
- code-based cryptography
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  - bright future!

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Finite fields are also widely used in

- coding theory
- algebraic geometry
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- coding theory
- algebraic geometry
- number theory
- motivates their study
  - **algorithmic** study: a part of **computer algebra**

Finite field arithmetic

Notation: $\mathbb{F}_{q^m}$ denotes *the* finite field with q^m elements

$$\mathbb{F}_{q^m} \cong \mathbb{F}_q[X]/(P(X))$$

- $P \in \mathbb{F}_q[X]$ is an **irreducible** polynomial of degree m

Some possible **representations**:

- **Zech's logarithm**: elements are represented as generator powers
- **normal** basis: $(\alpha, \alpha^{\sigma}, \dots, \alpha^{\sigma^{m-1}})$
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- **monomial** basis: $(1, \bar{X}, \dots, \bar{X}^{m-1})$
  - commonly used representation, easy to construct
  - multiplication slower than addition

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  - $O(m \log m)$ algorithm [Harvey, Van Der Hoeven '19]

Bilinear complexity: intuition

- $\mathcal{A}$ an algebra over $\mathbb{K}$
- **bilinear complexity:** number of subproduct in $\mathbb{K}$ needed to compute a product in $\mathcal{A}$

Karatsuba:

$$(a_0 + a_1 X)(b_0 + b_1 X) =$$

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Complexity of Karatsuba's algorithm

[redacted]

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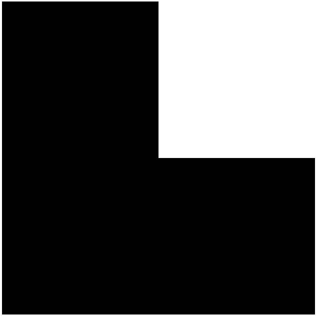

- Degree 2: 3 **multiplications instead of** 4

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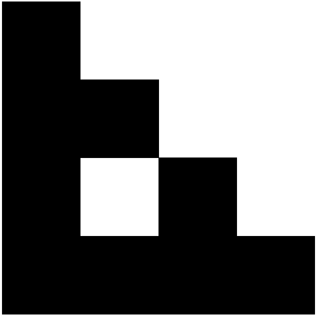

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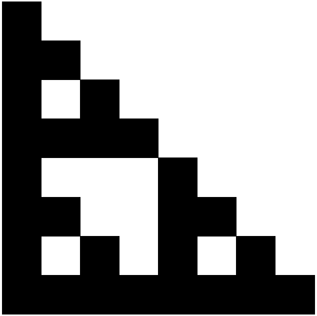

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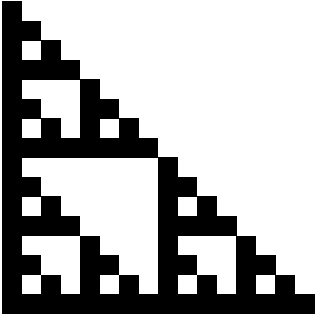

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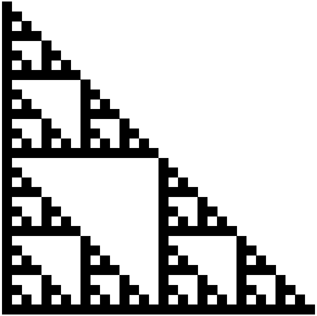

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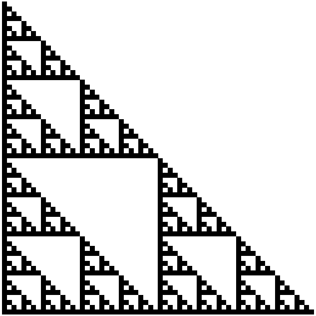

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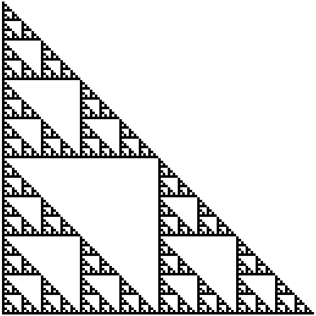

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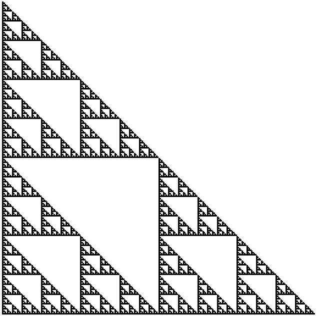

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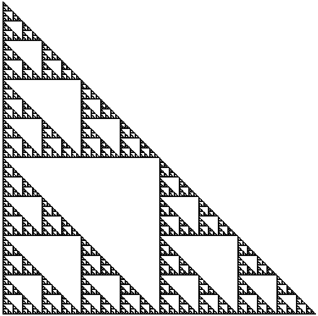

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2×2 matrix multiplication:

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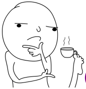

Open question: what is the bilinear complexity of the 3×3 matrix multiplication?

Bilinear complexity: definition

Definition

The **bilinear complexity** of the product in $\mathcal{A}$ is the minimal integer $r \in \mathbb{N}$ such that you can write, for all $x, y \in \mathcal{A}$

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Notations and questions

- $\mathbb{K} = \mathbb{F}_q$
- $\mu_q(m)$ = bilinear complexity of the product in $\mathcal{A} = \mathbb{F}_{q^m}$

Two independent questions:

- What is the asymptotic comportment of $\mu_q(m)$?
- Can we find values $\mu_q(m)$ for small m?

Asymptotics

Lower bound from coding theory

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Upper bounds, from **evaluation-interpolation** schemes

- [Chudnovsky-Chudnovsky '87]
- [Shparlinski-Tsfasman-Vladut '92]
- [Ballet '08]
- [Randriambololona '12]
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- $\mu_q(m)$ is **linear** in m

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- When studying $\mathcal{A} = \mathbb{F}_{q^m}$ for $m \to \infty$, one needs **many points** of evaluation

$\rightsquigarrow$ use a curve on $\mathbb{F}_q$ with many points of evaluation

HOW TO FIND SMALL VALUES?

Possibilities:

- tighten the theoretical bounds (hard ☹)
- find all formulas
  - clever **algorithms** for **exhaustive search**
  - [BDEZ '12]
  - [Covanov '18]

Symmetric decompositions

- $\mathcal{A}$ **commutative** algebra

Classic decompositions	**Symmetric** decompositions
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Notation: for $\mathcal{A} = \mathbb{F}_{q^m}$, we note $\mu_q^{\mathrm{sym}}(m)$ the minimal length r in a **symmetric** decomposition

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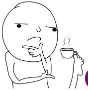

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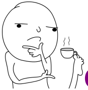

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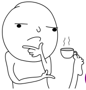

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Even more symmetric decompositions

- $\mathcal{A} = \mathbb{F}_{q^m}$
- every linear form φ can be written $x \mapsto \mathrm{Tr}(\alpha x)$ for some $\alpha \in \mathbb{F}_{q^m}$, with Tr the trace of $\mathbb{F}_{q^m}/\mathbb{F}_q$
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- we note $\mu_q^{\mathrm{tri}}(m)$ the minimal r in such formulas

Example of trisymmetric decomposition

- $\mathcal{A} = \mathbb{F}_{3^2} \cong \mathbb{F}_3[z]/(z^2 - z - 1) \cong \mathbb{F}_3(\zeta)$
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About trisymmetric decompositions

Link with other decompositions:

$$\mu_q(m) \leq \mu_q^{\mathrm{sym}}(m) \leq \mu_q^{\mathrm{tri}}(m)$$

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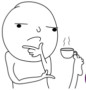

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Finding decompositions

Symmetric decompositions:

$$xy = \sum_{j=1}^{r} \mathrm{Tr}(\alpha_j x)\,\mathrm{Tr}(\alpha_j y) \cdot \beta_j$$

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- **ad hoc** algorithm

Computing trisymmetric decompositions

- choose a basis of $\mathbb{F}_{q^m}/\mathbb{F}_q$

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SOME RESULTS FOR $q = 3$

field	μ_q	μ_q^{sym}	μ_q^{tri}
$\mathbb{F}_{3^2}$	3	3	3
$\mathbb{F}_{3^3}$	6	6	6
$\mathbb{F}_{3^4}$	9	9	9
$\mathbb{F}_{3^5}$	$9 \leq \star \leq 11$	11	11
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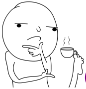

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Symmetry in higher dimensions

- What happens with the product of t variable $x_1, \ldots, x_t$, for $t \geq 3$?

Classic decompositions	**Symmetric** decompositions
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Proof.

Generalization of the Chudnovsky-Chudnovsky method: evaluation-interpolation on curves with many points. □

Back on trisymmetry

Corollary

Let $\mathcal{A} = \mathbb{F}_{q^m}$ and $q \geq 3$. Then the ***trisymmetric complexity*** $\mu_q^{tri}(m)$ *is* ***linear*** *in m.*

BACK ON TRISYMMETRY

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Proof.

Taking the trace on a **symmetric** decomposition for the 3 variable product xyz gives a **trisymmetric** decompositon for the product xy. □

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